Analysis 1, Summer 2023

List 1

Sequences, limits of sequences

20. If $a_n = (n+2)^3$, give the value of a_3.

21. For the sequence $b_n = n^{-n}$, what are the values b_1, b_2, and b_3?

22. If $c_n = (1+\frac{1}{n})^n$, what are the values c_1, c_2, and c_3? Give exact formulas (by hand) and decimal answers (using a calculator).

23. For the sequence $a_n = n^2 - 1$, give a formula for a_{n+1}.

24. Consider the sequence

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 22 \\ a_3 &= 222 \\ a_4 &= 2222 \\ a_n &= \underbrace{22...2}_{n \text{ digits}} \end{aligned}$$

   (a) Calculate $(10a_1+2) - a_1$, then $(10a_2+2) - a_2$, then $(10a_3+2) - a_3$.
   (b) Find a formula for $(10a_n+2) - a_n$ in terms of n only.
   (c) Find a formula for a_n.

> The sequence a_n **converges** to the real number L if for any $\varepsilon > 0$ there exists an N such that
> $$L - \varepsilon < a_n < L + \varepsilon \qquad \text{for all } n > N.$$
> In this case we say the **limit** of the sequence is L, and we write
> $$\lim_{n\to\infty} a_n = L.$$
> A sequence that does not converge to any number is said to **diverge**.

25. (a) For which positive integers n is $4 - \dfrac{1}{100} \;<\; \dfrac{8n}{2n+9} \;<\; 4 + \dfrac{1}{100}$?
   (b) For which positive integers n is $\dfrac{8n}{2n+9} \;=\; 4$?
   (c) Is it true that $\displaystyle\lim_{n\to\infty} \frac{8n}{2n+9} \;=\; 4$?

26. Calculate $\displaystyle\lim_{n\to\infty} \frac{3n^2 + n + \sqrt{n}}{5n^2}$.

27. Determine whether each sequence converges or diverges.
   (a) n^n
   (b) $\dfrac{n}{n+1}$
   (c) $(-1)^n$
   ☆(d) $\sin(3n)$
   (e) $\sin(\pi n)$
   (f) $\dfrac{(-1)^{n+1}}{n^n}$

We say a_n **diverges to infinity** and write $\lim_{n\to\infty} a_n = \infty$ if for any $M > 0$ there exist an N such that

$$a_n > M \qquad \text{for all } n > N.$$

Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ if for any $M > 0$ there exist an N such that

$$a_n < -M \qquad \text{for all } n > N.$$

28. Find the following limits if they exist.
    (a) $\lim_{n\to\infty} \frac{n+13}{n^2}$
    (b) $\lim_{n\to\infty} \frac{(n+5)(n-2)}{n^2-6n+7}$
    (c) $\lim_{n\to\infty} \frac{n^2}{n+13}$
    (d) $\lim_{n\to\infty} -2^n$
    (e) $\lim_{n\to\infty} (-2)^n$
    (f) $\lim_{n\to\infty} 2^{-n}$
    ☆(g) $\lim_{n\to\infty} 2^{1/n}$

29. Find $\lim_{n\to\infty} \big((9\sqrt{n} + \frac{1}{\sqrt{n}})^2 - 81n\big)$.

☆30. Find $\lim_{n\to\infty} n \cdot (2^{1/n} - 1)$. The ☆ means that this task is harder than what is normally expected in this course.

31. (a) Simplify the formula $\dfrac{\big(\sqrt{n} - \sqrt{n-1}\big)\big(\sqrt{n} + \sqrt{n-1}\big)}{\sqrt{n} + \sqrt{n-1}}$.
    (b) Find $\lim_{n\to\infty} \sqrt{n} - \sqrt{n-1}$.

32. Use the Squeeze Theorem with $\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ to find $\lim_{n\to\infty} \frac{\cos(n)}{n}$.

☆33. Use the fact that $\left(1 - \frac{1}{\sqrt{n}}\right)^n \leq \frac{1}{n}$ to find $\lim_{n\to\infty} (1/n)^{1/n}$.

34. (a) The *definition* of the number “0.385” is
    $$3 \cdot 10^{-1} \;\; + \;\; 8 \cdot 10^{-2} \;\; + \;\; 5 \cdot 10^{-2}.$$
    Write this number as a fraction (or an integer, if possible).
    (b) The *definition* of the number “0.2222...” is the ***limit*** of the sequence
    $$S_1 = 0.2$$
    $$S_2 = 0.22$$
    $$S_3 = 0.222$$
    $$S_4 = 0.2222$$
    $$S_n = 0.\underbrace{22...2}_{n \text{ digits}}$$
    Write this number as a fraction (or an integer, if possible).
    Hint: See Task 24(c).
    (c) The *definition* of the number “0.9999...” is the ***limit*** of the sequence
    $$S_n = 0.\underbrace{99...9}_{n \text{ digits}}.$$
    Write this number as a fraction (or an integer, if possible).